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Computability of Physical Operations

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It is shown (for a very simplified model) that a number-theoretic function representing an experimental physical setup is general recursive.

1. INTRODUCTION

Kreisel (1976) stated the following problem:

> We consider theories, by which we mean such things as classical or quantum mechanics, and ask if every sequence of natural numbers or every real number which is well defined (observable) according to the theory must be recursive or, more generally, recursive in the data (which, according to the theory, determine the observations considered). Equivalently, we may ask whether any such sequence of numbers, etc. can also be generated by an ideal computing or Turing machine if the data are used as input.

A similar problem, but with respect to human psychology and artificial intelligence, was investigated by Webb (1983). He formulated two theses:

(M) All human reasoning is a mechanical process (computation).
(C) Every "precisely described" piece of human behavior can be simulated by a suitable programmed computer.

In the theory of computability one formulates Church's thesis:

(CT) Every "effectively computable" function is general recursive (and vice versa).

Webb gives arguments that (CT) is within an eyelash of implying (C), which in turn provides, to the degree that experience confirms it, inductive support for (M). Moreover, he argues that Gödel incompleteness theorems

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provide an essential core of support for (CT) and that (CT) implies that any refutation of mechanism must employ noneffective constructions. So Gödel's work supports (M).

Feynman (1982) considered the possibility of exact simulation of physics by (a digital) computer, i.e., that the computer will do *exactly* the same as nature.

Primas (1972) gives heuristic arguments to support a straightforward physical interpretation of Church's thesis, i.e., that the channel of a constructable apparatus has to be a computable function in the sense of Turing.

In our work we show (for a very simplified model) that if a number-theoretic function f represents a deterministic experimental physical setup, then the function is a general recursive one.

2. FORMALISM

We give an outline of a formalized empirical physical theory of an apparently trivial kind. The theory is set up by analogy with a first-order mathematical theory. In the language of our theory we would like to describe physical experiments understood in a very general sense. A real model of the theory could be, e.g., a transmitter (a physical operation), i.e., a macroscopic object with one input and one output channel. A system prepared in some state enters the input channel and after transformation exits the output channel. By a transmitter we also can understand the action of external fields. So we can have either an approximately instantaneous change of the system or a more or less continuous evolution. After the transmission process we investigate the properties of the output system. As another model one could take a measuring apparatus.

Being a physical, our theory should consist of a mathematical structure (syntax) together with a set of rules of interpretation (semantics).

It should be stated clearly that what is presented in this paper is not a realistic picture, but a simplified and idealized schema of the subject under investigation.

2.1. Syntax

From a syntactical point of view there is no essential difference between an empirical theory and a first-order mathematical one. The formalized language of our theory will comprise signs for natural numbers $1, 2, \ldots, k, n, \ldots$ and one *functional constant* f denoting a number-theoretic function. *Terms* are of the form $f(4)$, $f(k)$, etc. [In some cases we can take as terms also expressions like $fff(12)$, etc].

The *atomic formulas* are identities between terms and *compound formulas* are formed from the atomic ones by means of the connectives of the propositional calculus.

2.2. Semantics

Our theory is an empirical one and therefore we assume that formulas can be proved or disproved by means of (finite) operational tests.

How can one look at a physical experiment? We have a material device (represented in our language by the functional constant f)—a certain selected portion of the physical universe—an experimental setup. Natural numbers $1, 2, \ldots$ denote input-output conditions (pre-postcontrolling conditions), which can be operationally tested. We assume as usual that there is only a countable number of properties of a physical system.

Thus, the symbol "k" may be regarded as a "label" which is attached to a *document* giving a description of a construction of input conditions with property "k". We provide input conditions "k", f runs for a certain time; and then we observe the output conditions "l".

We assume that our experiments are deterministic ones in the Daniel-Gisin (1983) sense, i.e., for one input condition we obtain at most one output. Therefore the functional constant f denoting our experimental setup is a number-theoretic function.

Now the question arises of whether the function is a computable (general recursive) one.

How can one determine the properties of our experimental setup f? The setup is a material object and therefore it conforms to the laws of some physical theory. Thus, some of its properties can be deduced *a priori* from laws and theorems of the theory. Formulas determining these properties form a set of *theoretical statements or meaning postulates.* Moreover, performing experiments by means of the setup f, we obtain another of its properties determined in the language of our theory by formulas which form a set of *observation statements.*

It is rather obvious that direct observation presents the basic noninferential method of validation in empirical science. Observation statements are just sentences capable of being validated by direct observation. One can validate such a statement without resorting to any inference—by simply observing the objects this statement is about. Thus, the observational statements lie at the foundations of the whole of scientific knowledge; whatever is asserted by the scientist is either expressed in observation statements, or has been validated by being inferred from observation statements.

The language L of our theory of the experimental setup f will simply be identified with the set of all its (theoretical and observational) formulas.

Moreover, we assume that the axioms of the classical propositional calculus and the identity calculus are valid, together with every formula

$$n_1 \neq n_2$$

n_1, n_2 being distinct numerals; as a sole rule of inference we take *modus ponens*, i.e., if α, β are formulas and $\rightarrow$ denotes the operation of logical implication then "α", "$\alpha \rightarrow \beta$" true implies "β" true. In all languages of classical and quantum theories both *modus ponens* and *modus tollens* remain valid forms of arguments. (Hardegree, 1979).

Modus ponens as a rule of inference is a syntactical characterization of the operation of logical consequence in L. The operation is characterized as follows: α is a *logical consequence* of X (X is a set of formulas), "$\alpha \in \mathrm{Cn}(X)$," iff there is a proof of α from X (the concept of proof is defined along the usual lines).

A *theory* then always comprises all of its logical consequences: $\mathrm{Cn}(T) \subseteq T$; it is thus what logicians call a *system* (*of logic* Σ).

The function f is called *a model* of a theory T if f satisfies each formula belonging to T. In this way we determine a syntax of a very simple empirical theory T. The fundamental problem here concerns the distinction between the empirical and *a priori* elements inherent in any such theory. We assume that *a priori* (and of course empirical) elements in our theory are in the form of finitely testable formulas, i.e., one can in a finite number of steps determine whether a formula is true or false.

A theory T is always an infinite set of statements. We can distinguish two cases: A theory for which there exists an effective procedure enabling anyone to decide in a finite number of predetermined steps whether or not any given formula in L is a theorem of the theory is called *decidable*; one that does not satisfy this condition is *undecidable*.

Definition (Przełęcki, 1969). A theory T is said to be *axiomatizable* if all its theorems follow from a decidable subset of them, that is, if there is a decidable set A of formulas, called the set of axioms, such that $T = \mathrm{Cn}(A)$.

Axiomatization of a theory concerns only its formal presentation; it does not presuppose anything with regard to its interpretation and validation. Because the known actual empirical theories are axiomatizable, we assume the same of our simplified theory of the experimental setup.

Definition. A function φ is called a *model* of a set Σ of finite formulas if φ satisfies each formula belonging to Σ.

It is not sufficient to consider satisfaction by number-theoretic functions only. There are many sets of finite formulas which are satisfiable by some function, but not by any number-theoretic function. If the function φ from the above definition is a number-theoretic one, then the model is called *standard.*

Definition. A theory T is called *categorical* if it has exactly one standard model.

We assume that our set of theoretical and empirical statements unambiguously determines the experimental setup f, so it is a categorical system.

Definition. A theory T is *complete* if for each formula α, $\alpha \in T$ or non-$\alpha \in T$.

Definition. A theory T is called *consistent* if, for no formula α, both formulas α and non-α are theorems in T.

Theorem (Gödel). If a set of (finite) formulas is consistent, it has a model.

Now we will prove the main theorem of our work.

Theorem. (1) If an (empirical) theory T is categorical and has no nonstandard models, then it is complete. (2) If in addition it is axiomatizable, then it is decidable and the unique function that satisfies it is recursive.

Proof. (1) The theory T has only one standard model and no nonstandard ones, thus it has exactly one model. Suppose that T is not complete. Then there is a formula α such that neither α nor non-α belongs to T. Now, T is a theory, i.e., a deductively closed system, and therefore neither α not non-α can be deducible from formulas belonging to T. Therefore each of the sets of formulas $T+\{\alpha\}$ and $T+\{\text{non-}\alpha\}$ is consistent and by the Gödel theorem it has a model. So there are two distinct models of T; contradiction.

(2) Let T be categorical; hence, it has a model and therefore is consistent. Let T have no nonstandard models and let it be axiomatizable. By part 1, T is complete, and from consistency it follows that for each formula α either α or non- α (not both) belongs to T. The set of theorems of axiomatizable theory is recursively enumerable. Hence and from completeness it follows that for any formula α it is possible to determine (in an effective way) whether α or non-α belongs to T. So T is decidable. Let m be any integer. Because T has no nonstandard models and is consistent, there is exactly one n such that formula $f(m)=n$ belongs to T. The theory T is decidable and therefore this n can be effectively found from m by merely generating T until the formula appears. Thus, if we accept Church's thesis, f is a recursive function.

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